Doctoraat Bram De Rock: The Anosov relation for Nielsen numbers of maps of infra–nilmanifolds

After graduating from the secondary school OLVA in Brugge, I studied mathematics at the K.U.Leuven. From October 2000 till September 2006, I was an assistant at the Campus Kortrijk. Under guidance of prof. dr. Karel Dekimpe and prof. dr. Wim Malfait, I prepared my PhD thesis, which was defended on February 20, 2006.

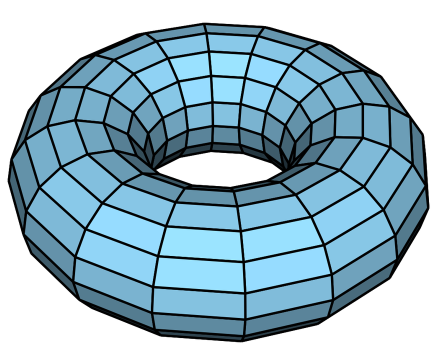
In 1985, D. Anosov proved in [2] that any map $f : M \to M$ of a nilmanifold M, always satisfies the Anosov relation, i.e. $N(f) = |L(f)|$. This gives us an easy method to compute the Nielsen number $N(f)$ via the Lefschetz number $L(f)$. In literature, one can find many generalizations of this result and in my PhD thesis, we extend this result towards maps of infra-nilmanifolds in two ways. In a first, more natural, way we look for classes of infra-nilmanifolds for which the same result holds. For the second way, we work with classes of maps on infra-nilmanifolds instead of considering all maps.

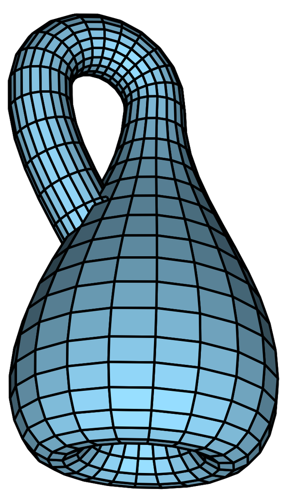
Introduction and Motivation

Infra–nilmanifolds

Let G be a connected, simply connected, nilpotent Lie group with Lie algebra $\mathfrak{g}$. An affine endomorphism of G is an element $(\delta, \mathfrak{D})$ of the semigroup $G \rtimes \mathrm{Endo}(G)$ with $\delta \in G$ the translational part and $\mathfrak{D} \in \mathrm{Endo}(G)$ the linear part. If the linear part $\mathfrak{D}$ belongs to $\mathrm{Aut}(G)$, then $(\delta, \mathfrak{D}) \in \mathrm{Aff}(G) = G \rtimes \mathrm{Aut}(G)$.

An almost-crystallographic group is a subgroup E of $\mathrm{Aff}(G)$ such that its subgroup of pure translations $N = E \cap G$, is an uniform lattice of G and moreover, N is of finite index in E. Therefore the quotient group $F = E/N$ is finite and is called the holonomy group of E. When E is a torsion-free almost-crystallographic group, the orbit space $M = E \backslash G$ is called an infra-nilmanifold. In the special case that $E = N$, M is called a nilmanifold and when G is abelian, i.e. $G = \mathbb{R}^n$, M is a flat manifold. Below we present two examples of flat manifolds: on the left the torus (a nilmanifold) and on the right the Klein bottle (an infra-nilmanifold).





Any almost-crystallographic group determines a faithful representation $T : F \to \mathrm{Aut}(G)$, which is referred to as the holonomy representation. and by taking differentials, this also induces a faithful representation $T_* : F \to \mathrm{Aut}(\mathfrak{g}) : x \mapsto T_*(x) = d(T(x))$.
Finally, in [1], K.B. Lee proves the following result concerning maps of infra-nilmanifolds.

Theorem *Let $M = E \backslash G$ be an infra-nilmanifold and $f : M \to M$ a map of M. Then f is homotopic to a map $h : M \to M$ induced by an affine endomorphism $(\delta, \mathfrak{D}) : G \to G$.*

Fixed Point Theory

Let $f : X \to X$ be a map of a compact connected manifold and denote the fixed point set of f by $\mathrm{Fix}(f) = \{x \in X \mid f(x) = x\}$. One of the main objectives in fixed point theory is to calculate $MF(f)$, the minimum number of fixed points among all maps homotopic to f. In principle, to calculate $MF(f)$ it is necessary to examine the fixed point set of every map homotopic to f. To avoid this, several numbers associated to f are defined to provide information on $MF(f)$.
A first number, is the Lefschetz number

$$L(f) = \sum_i (-1)^i \ \mathrm{Trace}(f_* : H_i(M, \mathbb{Q}) \to H_i(M, \mathbb{Q})).$$

The Lefschetz number is a (reasonably) computable invariant of f but does not give a lot of information on the fixed points of f.
A second number is the Nielsen number $N(f)$ which is an (almost always sharp) lower bound for $MF(f)$ and for infra-nilmanifolds $MF(f) = N(f)$. To define $N(f)$, we need the concept of essential fixed point classes. Therefore we partition $\mathrm{Fix}(f)$ into equivalence classes, referred to as fixed point classes, by the relation: $x, y \in \mathrm{Fix}(f)$ are f-equivalent if and only if there is a path w from x to y such that w and fw are homotopic. Fixed point classes which persist under all homotopies are called essential fixed point classes and

$$N(f) = \text{ the number of essential fixed point classes of } f.$$

As $MF(f)$, $N(f)$ is also not readily computable. Therefore, a possible way to obtain information on $N(f)$, is to show that f satisfies the Anosov relation which is the subject of my thesis. To do so, we use the following result of K.B. Lee ([1]).

Theorem *Let $f : M \to M$ be a map of an infra-nilmanifold M with homotopy lift $(\delta, \mathfrak{D})$. Let $T : F \to \mathrm{Aut}(G)$ be the holonomy representation. Then $N(f) = L(f) \Leftrightarrow \forall x \in F : \det(I_n - T_*(x)\mathfrak{D}_*) \geq 0$ and $N(f) = -L(f) \Leftrightarrow \forall x \in F : \det(I_n - T_*(x)\mathfrak{D}_*) \leq 0$.*

The Results

Classes of infra–nilmanifolds

Let M be an n-dimensional infra-nilmanifold with holonomy group F. Based on F we distinguish three classes of infra-nilmanifolds.

1. The order of F is odd.
2. F is a cyclic group.
3. $F = \mathbb{Z}_2^{n-1}$ (and M is called a flat generalized Hantzsche-Wendt manifold).

Theorem *If M is an infra-nilmanifold with odd order holonomy group F, then $N(f) = |L(f)|$ for any map $f : M \to M$.*

Theorem *Let M be an infra-nilmanifold with cyclic holonomy group F generated by x_0. Let $T : F \to \mathrm{Aut}(G)$ be the holonomy representation and suppose -1 is not an eigenvalue of $T_*(x_0)$. Then for any map $f : M \to M$ we have that $N(f) = |L(f)|$.*

Theorem *Let $n \geq 3$ be an odd integer and M a (flat) orientable n-dimensional generalized Hantzsche-Wendt manifold. Then for each map $f : M \to M$ we have that $N(f) = |L(f)|$.*

Classes of maps

Let $f : M \to M$ be a map on an infra-nilmanifold M with homotopy lift $(\delta, \mathfrak{D})$. Based on the eigenvalues λ_i of $\mathfrak{D}_*$ we can then distinguish three types of maps.

1. Expanding maps: $|\lambda_i| > 1$ for all i.
2. Nowhere expanding maps: $|\lambda_i| \leq 1$ for all i.
3. Anosov diffeomorphisms: $|\lambda_i| \neq 1$ for all i and $\prod_i \lambda_i = 1$.

Theorem *Suppose $f : M \to M$ is an expanding map of an infra-nilmanifold M. Then $N(f) = |L(f)|$ if and only if M is orientable.*

Theorem *Let $f : M \to M$ be a nowhere expanding map on an infra-nilmanifold M, then $N(f) = L(f)$.*

The results for the third type can be loosely summarized as that there is not much correlation between Anosov diffeomorphisms and the Anosov relation.

Small dimensions

We also examine what we already know about the Anosov theorem in low dimensions, i.e. up to dimension 4. For all the manifolds which are not covered by the theoretic results, we present a counterexample (in the case the Anosov theorem does not hold) or a proof (in the opposite case). Because of the lack of space, I only want to mention the following intriguing questions which are motivated by the results in the low dimensions.

Question *Does the Anosov theorem hold for any (flat) manifold with holonomy group A_4?*

Question *Can we capture the relationship between the structure of the non-abelian Lie group G and the validity of the Anosov theorem? More specific, what if we restrict ourselves to infra-nilmanifolds with non-abelian holonomy group?*

Referenties

[1] Lee, Kyung Bai, *Maps of infra-nilmanifolds* PACJM, volume 168, No. 1, 1995, pp. 157–166.

[2] Anosov, D.V., *The Nielsen numbers of maps of nilmanifolds*,Uspekhi. Mat. Nauk, number 40, 1985, pp. 133–134, English transl.: Russian Math. Surveys, 40 (no. 4), 1985, pp. 149–150.

[3] De Rock, Bram, *The Anosov relation for Nielsen numbers of maps of infra-nilmanifolds*, 2006.